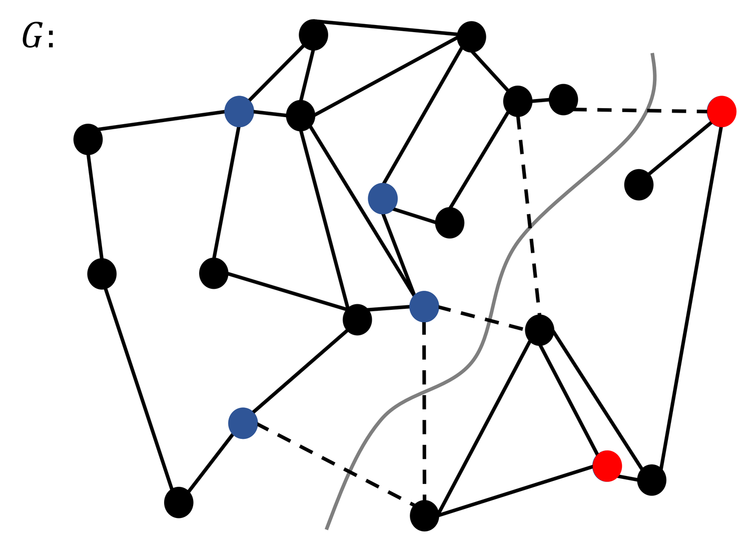
מכון ויצמן למדע
WEIZMANN INSTITUTE OF SCIENCE

Refined Vertex Sparsifiers of Planar Graphs

Robert Krauthgamer, Inbal Rika
Weizmann Institute of Science, Rehovot, Israel
robert.krauthgamer@weizmann.ac.il, inbal.rika@weizmann.ac.il

Cut Sparsifiers

Let G be an **undirected network** with edge capacities $c: E(G) \to R^+$ and k terminals $T \subseteq V(G)$.



We care about **terminal cuts**:

$mincut_G(S)$ = minimum-capacity cut separating $S \subset T$ and $\bar{S} = T \setminus S$.

Definition 1. A network H is a (q, s)***-cut sparsifier*** (quality, size) of G if $|V(H)| \le s$ and $\forall\ S \subset T, \mathrm{mincut}_G(S) \le \mathrm{mincut}_H(S) \le q \cdot \mathrm{mincut}_G(S)$.

Question 2. What is the **best tradeoff** between the quality q and the size s of (q, s)-cut sparsifier for k-terminal networks?

Definition 3. A ***mimicking network*** is a cut sparsifier of quality $q = 1$, ***i.e.*** $\forall S \subset T, \mathrm{mincut}_H(S) = \mathrm{mincut}_G(S)$.

Question 4. What is the **smallest** mimicking network size for every k-terminal network G?

For a planar k-terminal network G, let $\gamma(G)$ be the minimum number of faces that are incident to all the terminals of G.

Known and New Bounds for Mimicking Networks

Graphs	Size	minor	Reference
General	2^{2^k}	No	[HKNR98, KR14]
Planar	$O(k^2 2^{2k})$	Yes	[KR13]
Planar	$O(k 2^{2k})$	Yes	New
Planar $\gamma = \gamma(G)$	$O(\gamma 2^{2\gamma} k^4)$	Yes	New

References.

[HKNR98] T. Hagerup, J. Katajainen, N. Nishimura, and P. Ragde. Characterizing multiterminal flow networks and computing flows in networks of small treewidth.

[KR13] R. Krauthgamer and I. Rika. Mimicking networks and succinct representations of terminal cuts.

[KR14] A. Khan and P. Raghavendra. On mimicking networks representing minimum terminal cuts.

Elementary Cutsets

For $S \subset T$, let $E_S \subseteq E(G)$ be the cutset that separates between S and $\bar{S}$ in G with minimum capacity, i.e. $c(E_S) = mincut_G(S)$.

Definition 5. A minimum cutset E_S is called an ***elementary cutset*** if $G \setminus E_S$ has exactly 2 connected components.

$$\mathcal{T}_e(G) := \{S \subset T \mid E_S \text{ is an elementary cutset}\}.$$

Theorem 6. Every minimum cutset E_S can be decomposed into a disjoint union of elementary cutsets, i.e. $\exists \phi \subset \mathcal{T}_e(G)$ such that $E_S = \cup_{S' \in \phi} E_{S'}$.

We show that only elementary cutsets matter.

Theorem 7. Let G, H be networks with the same terminals T. If $\mathcal{T}_e(G) = \mathcal{T}_e(H)$, and $\forall S \in \mathcal{T}_e(G), \mathrm{mincut}_G(S) \le \mathrm{mincut}_H(S) \le q \cdot \mathrm{mincut}_G(S)$, then H is a cut sparsifier of quality q of G.

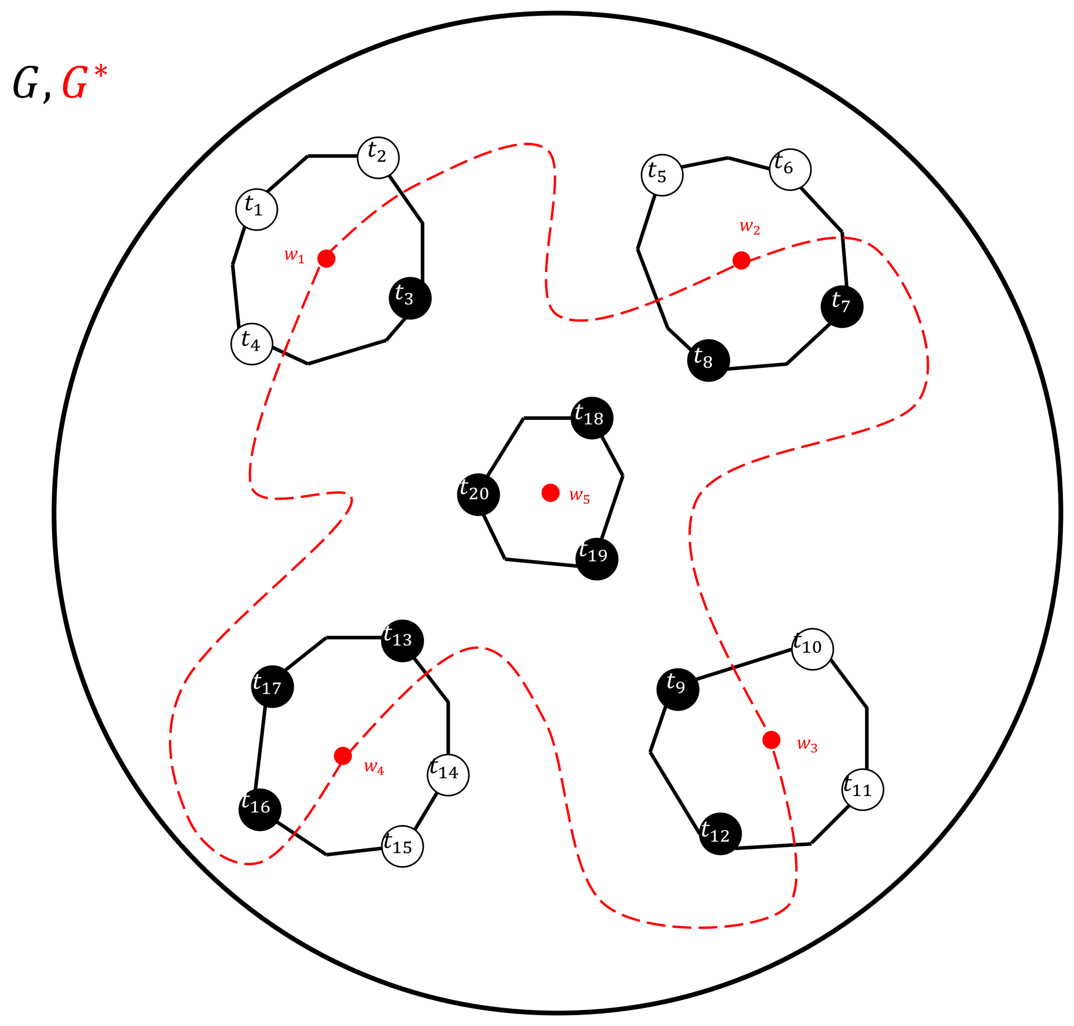
Improved Mimicking Networks for Planar Graphs

Theorem 8. $\forall$ planar k-terminal network G with $\gamma = \gamma(G)$ $\exists$ p $= O(2^\gamma k^2)$ subsets of edges $E_1, \dots, E_p \subset E$, such that every elementary cutset E_S in G can be decomposed into a disjoint union of these E_i's.

Proof Idea.

- Elementary cutset E_S in G ➔ E_S^* simple cycle in dual G^*.
- Decompose E_S^* into simple paths $P_1 \dots P_l$.
- Characterize each P_i independently of E_S^*.
- Bound the number of different P_i by f(γ) instead of f(k).

Theorem 9. $\forall$ planar k-terminal network G with $\gamma = \gamma(G)$ $\exists$ a minor mimicking network of size $O(\gamma\ 2^{2\gamma} k^4)$.



Theorem 10. $\forall$ planar k-terminal network G, such that $\forall S, S' \in \mathcal{T}_e(G)$ the graph $G \setminus (E_S \cup E_{S'})$ has at most α connected components, $\exists$ a minor mimicking network H of size $O(\alpha \cdot |\mathcal{T}_e(G)|^2)$.

$\forall$planar k-terminal network G, $\alpha \le k$ and $|\mathcal{T}_e(G)| \le 2^k$

Corollary 11. $\forall$ planar k-terminal network G $\exists$ a minor mimicking network of size $O(k 2^{2k})$.

Terminal-Cuts Scheme

Definition 12. A ***terminal-cuts scheme*** (TC-scheme) is a data structure that support the following two operations on a k-terminal network G of size n and $c: E \to \{1, \dots, n^{O(1)}\}$.

- *Preprocessing.* Which gets G, and builds storage (memory) M.
- *Query.* Which gets $S \subset T$, and uses M to output $\mathrm{mincut}_G(S)$.

Theorem 13. $\forall$ k-terminal network G $\exists$ a TC-scheme with size(M)$\le O(|\mathcal{T}_e(G)|(k + \log n))$ bits.

Theorem 14. $\forall$ planar k-terminal network G with $\gamma = \gamma(G)$ $\exists$ a TC-scheme with size(M)$\le O(2^\gamma k^2(\gamma + \log n))$ bits.

In addition:

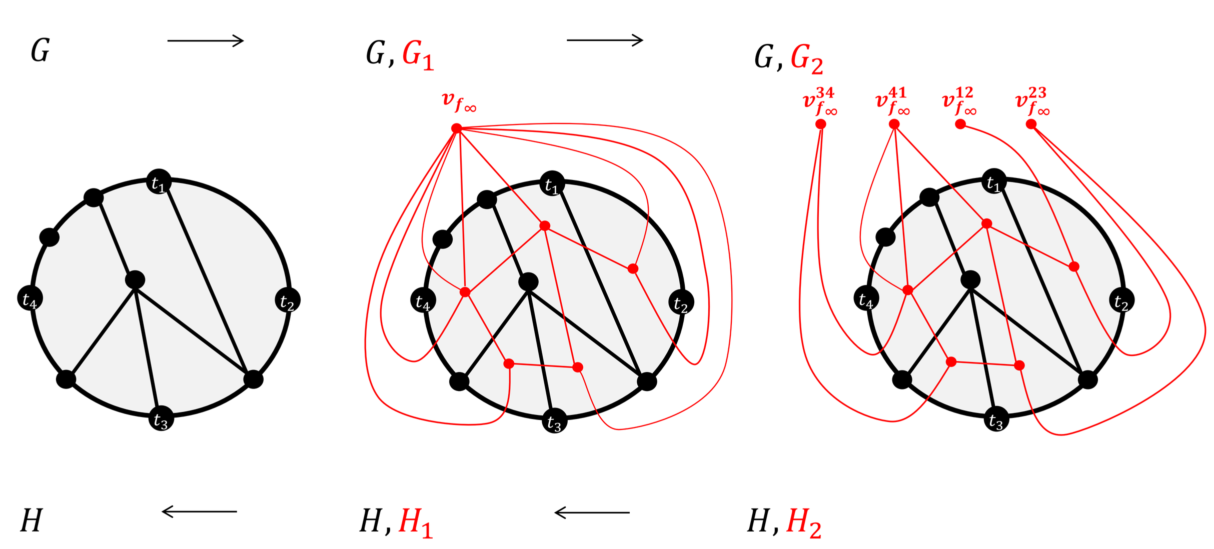
Equivalence Between Cut and Distance Sparsifiers.

Definition 15. A network H is called a (q, s) ***–Distance Approximating Minor (DAM)*** of G, if H is a minor of G, $|V(H)| \le s$ and $\forall\ t, t' \in T,\ d_G(t, t') \le d_H(t, t') \le q \cdot d_G(t, t')$.

Theorem 16. Let G be a planar k-terminal network with $\gamma(G) = 1$ and with edge-capacities.

One can construct a planar k-terminal network G' with $\gamma(G) = 1$ and with edge-lengths, such that

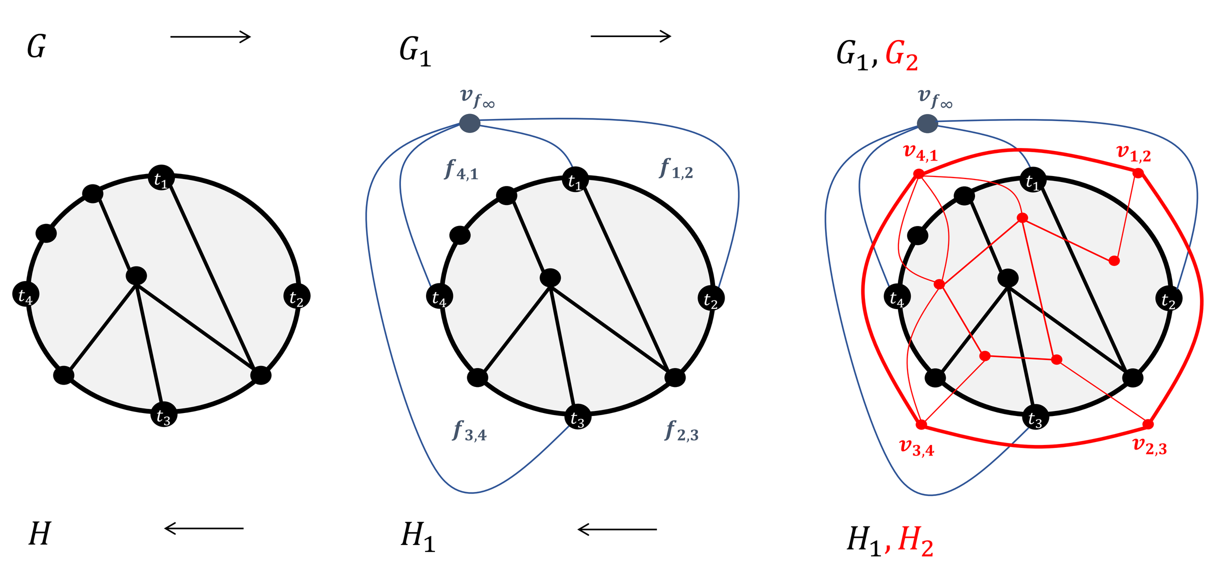
G' admits a (q, s)-DAM ➔ G admits a minor $(q, O(s))$-cut sparsifier.



Theorem 17. Let G be a planar k-terminal network with $\gamma(G) = 1$ and with edge-lengths.

One can construct a planar k-terminal network G' with $\gamma(G) = 1$ and with edge-capacities, such that

G' admits a minor $(q, O(s))$-cut sparsifier ➔ G admits a (q, s)-DAM.



Consequently, the same (q, s) bounds hold for distance sparsifiers and for cut sparsifiers.