Online Chromatic Number is PSPACE-Complete

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Online Graph Coloring

- There is an undirected graph G known in advance.
- Vertices arrive one by one in an unknown order.
- An online algorithm must immediately and irrevocably assign a color to each incoming vertex v so that the revealed graph is properly colored.
- The exact location of v in G is not known, the algorithm only sees edges to previously colored vertices.

Definition (Online chromatic number $\chi^O(G)$)

$\chi^O(G)$ is the smallest k s.t. there exists a deterministic algorithm which is able to color G using k colors for any incoming order of vertices.

- Deciding $\chi^O(G) \leq k$ is in PSPACE and coNP-hard [Kudahl '14].
- Conjecture: PSPACE-hard [Kudahl '14].

Main theorem

We resolve the complexity of computing $\chi^O(G)$:

Theorem

Deciding $\chi^O(G) \leq k$ is PSPACE-complete.

Game view

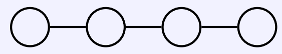
- Two players: Drawer and Painter
- At each round:
  - Drawer (the adversary) chooses an uncolored vertex v and sends it to Painter without any information to which vertex of G it corresponds, only revealing the edges to the previously sent vertices.
  - Painter (the online algorithm) must color v properly (it cannot use a color of a neighbor of v).
- Asymmetric game: Drawer has much more control than Painter
- In most PSPACE-complete games players have roughly the same power.

Deciding $\chi^O(G) \leq k$ is in PSPACE

Game tree evaluation using Minimax in poly-space:

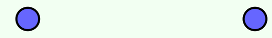
- # of rounds is at most $|V(G)|$.
- Painter just tries at most $|V(G)|$ colors.
- Drawer has at most 2^s moves where s = # of colored vertices, since it chooses which colored vertices shall be adjacent to the incoming vertex. Sent vertices must form an induced subgraph of G which can be checked in poly-space.

Example: $\chi^O(P_4) = 3$

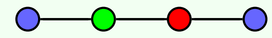


First, Drawer sends two non-adjacent vertices:

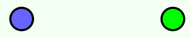
The same color case



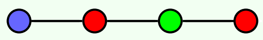
Then vertices are the endpoints of P_4:



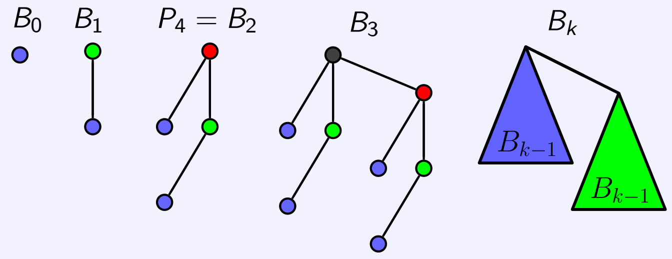
Different colors case



Then one vertex is the 1st and the other is the 3rd vertex of P_4:



Binomial trees



$\chi^O(B_k) = k+1$ [Gyárfás, Lehel '90] (by a similar strategy as for P_4).

PSPACE-hardness

Q3DNF-SAT

The satisfiability of a fully quantified formula Q in the 3-disjunctive normal form (3-DNF) such as

$\forall x_1 \exists x_2 \forall x_3 \exists x_4 \ldots : (x_1 \wedge x_2 \wedge \neg x_3) \vee (\neg x_1 \wedge x_2 \wedge \neg x_4) \vee \ldots$

PSPACE-complete (triv. reduction from Q3CNF-SAT).

Precoloring

- G_p = subgraph precolored before the game between Drawer and Painter.
- Drawer also reveals edges to G_p for each incoming vertex.
- Deciding $\chi^O(G) \leq k$ for G with precoloring is PSPACE-complete [Kudahl '14] (reduction from Q3DNF-SAT).
- Intuitively, precoloring gives some advantage to Painter.

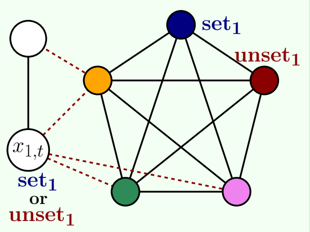
Step 1. Large precoloring

Given a formula Q, we create a graph G_1 which will simulate the formula:

$\chi^O(G_1) \leq k_1$ for some k_1 iff Q is satisfiable.

The gadgets:

Precolored clique K_{col}

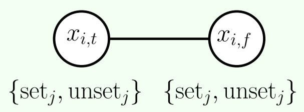


- Contains all k_1 colors.
- Colors have names, e.g., set$_i$, f_a, false$_a$.
- Each vertex $v \notin K_{col}$ has a set of allowed colors.
- v is connected to all not allowed colors in K_{col} (red edges).

Blue edges

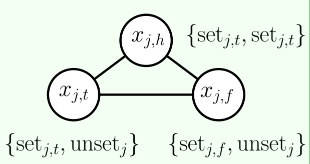
Help Painter distinguish vertices $x_{i,t}$ and $x_{i,f}$ for $\forall x_i$. Thus Drawer must first send variable vertices and then other vertices.

Gadget for $\forall x_i$



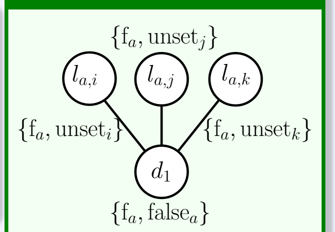
Painter *cannot* distinguish $x_{i,t}$ and $x_{i,f}$ if there are no edges to other vertices.

Gadget for $\exists x_j$



Painter *can* distinguish $x_{j,t}$ and $x_{j,f}$.

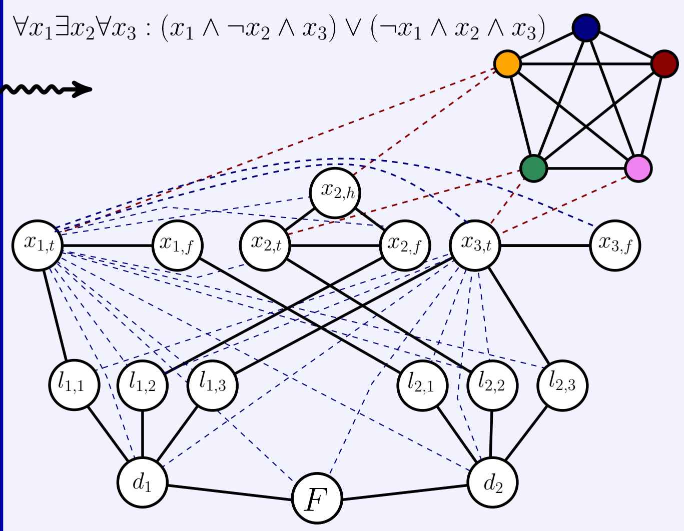
Gadget for clause a



Final vertex F

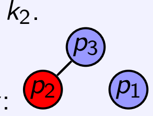
F can be colored with a color false$_a$ iff the formula is satisfiable.

Step 1. Big picture of G_1

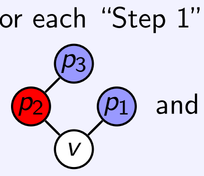


Step 2. Log. many precolored vertices

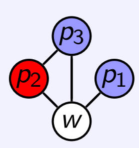
- Given G_1 we construct G_2 s.t.
  $\chi^O(G_1) \leq k_1$ iff $\chi^O(G_2) \leq k_2$.
- K_{col} not precolored, but present.
- One node for each "Step 1" vertex v:



- Edges to v:



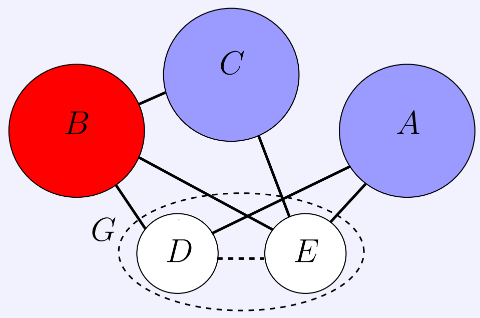
and to $w \neq v$:



- $x_{i,t}$ and $x_{i,f}$ for $\forall x_i$ identified by the same two nodes.
- $\mathcal{O}(\log n)$ precolored vertices to distinguish n nodes (binary encoding).
- A node arrives early $\Rightarrow$ can be used for recognition.
- Arrives after gadgets: Painter saves a color.

Step 3. Removing a precolored vertex

- Given G with p precolored vertices we create G' with $p-1$ precolored vertices such that:
  - $\chi^O(G) \leq k$ iff $\chi^O(G') \leq k'$,
  - $|V(G')| \leq 25 \cdot |V(G)|$.
- We replace a precolored vertex v_p in G by a "supernode" with three huge cliques A, B, C:



- D = nonprecolored part of G not connected to v_p.
- E = nonprecolored part of G connected to v_p.
- Solid edge = complete bipartite graph between two parts.
- Dashed edge = edges between D and E as in G.
- The sheer size of the supernode allows Painter to use it like a precolored vertex, or to save many colors if it does not arrive early.

Conclusions

- Applying Step 3 log. many times on G_2 yields a graph G_3 with no precolored vertex s.t.
  $\chi^O(G_3) \leq k_3$ iff the formula Q is satisfiable.
- $|V(G_3)|$ is polynomial in the size of Q.
- Since all constructions run in polynomial time, this proves the theorem. □

Reference

M. Böhm, P. Veselý: Online Chromatic Number is PSPACE-Complete. In *Proc. of the 27th International Workshop on Combinatorial Algorithms (IWOCA)*. LNCS 9843, 16–28 (2016). Best Student Paper of IWOCA 2016.

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