Succinct Progress Measures and Solving Parity Games in Quasi-Polynomial Time

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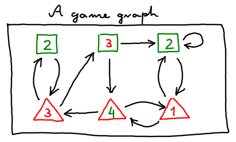
RELEVANT RECENT PAPERS

- Calude, Jain, Khoussainov, Li, Stephan — STOC 2017 — Solving parity games in quasipolynomial time — Best Paper Award
- J., Lazić — LICS 2017 — Succinct progress measures for solving parity games
- Gimbert, Ibsen-Jensen — arXiv 1702 — A short proof of correctness of the quasi-polynomial time algorithm for parity games
- Fearnley, Jain, Schewe, Stephan, Wojtczak — SPIN 2017 — An ordered approach to solving parity games in quasipolynomial time and quasilinear space

PARITY GAMES

$n = |V|$
$m = |E|$

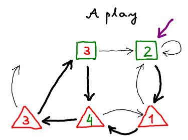
A game graph



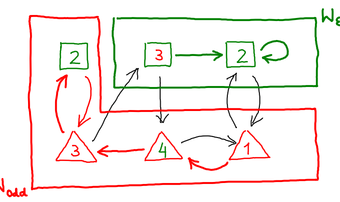
$G = (V = V_{Even} \uplus V_{Odd}, E, \pi)$

$\pi : V \to \{1,2,3,4,5,\ldots,d\}$

A play



POSITIONAL DETERMINACY



Corollary: Deciding the winner in parity games is in NP ∩ co-NP

APPLICATIONS OF PARITY GAMES

- Automata theory
  - complementation
  - emptiness
  - translations
- Logic
  - satisfiability
  - fixpoint logics
- Verification
  - model checking
  - fair (bi)simulation
  - program analysis and repair
- Synthesis
- Databases and XML

- pushdown graphs
- hierarchical structures
- higher-order recursion schemes
- universal coalgebra
- stochastic systems
- timed systems
- hybrid systems

WIDER IMPACT OF PARITY GAMES

- Structural graph theory for directed graphs
- Time complexity of Howard's policy iteration
- Time complexity of (randomized) simplex pivoting rules
- Computational complexity of search problems
- Computational complexity of path-following algorithms

COMPLEXITY OF DIVIDE-AND-CONQUER ALGORITHMS

- Plain vanilla [McN'93, Zie'98]: $n^{d+O(1)}$
  $\boxed{T(n,d) \leq n \cdot T(n,d-1) + O(nm)}$
- Dominion preprocessing by brute force [JPZ'06'08]: $n^{O(\sqrt{n})}$
  $\boxed{T(n) \leq T(n-1) + T(n-\sqrt{n}) + n^{O(\sqrt{n})}}$
- Dominion preprocessing by progress measure lifting [Sch'07'17]:
  $\boxed{T(n,d) \leq \sqrt{n} \cdot T(n,d-1) + n^{\frac{d}{3}+O(1)}}$
  $n^{\frac{d}{3}+O(1)}$

AN "ADAPTIVE" ORDER ON $\mathbb{B}^*$

For all $b \in \mathbb{B}$ and $s, t \in \mathbb{B}^*$:

- $0s < \varepsilon$
- $\varepsilon < 1s$
- $bs < bt$ iff $s < t$

$\boxed{0 < \varepsilon < 1}$

where $\varepsilon \in \mathbb{B}^*$ is the empty string

A succinct PROGRESS MEASURE

$\boxed{\mu : V \to \mathbb{N}^{d/2}}$ $\mathbb{L}_{n,d/2}$

- Edge (v,u) is progressive if
  - $\mu(v)|_{\pi(v)} \succcurlyeq_{lex} \mu(u)|_{\pi(v)}$ and $\pi(v)$ is even
  - $\mu(v)|_{\pi(v)} \succ_{lex} \mu(u)|_{\pi(v)}$ and $\pi(v)$ is odd
- Vertex v is progressive if
  - $v \in V_{Even}$ and at least one outedge is progressive
  - $v \in V_{Odd}$ and all outedges are progressive

$\boxed{\mu \text{ is a succinct progress measure if all vertices are progressive}}$

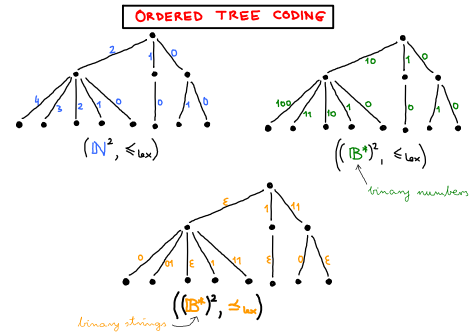
THE SUCCINCT PROGRESS MEASURE THEOREM

Thm TFAE

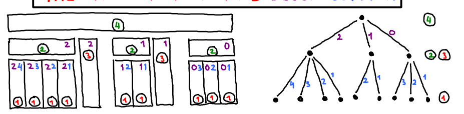
(1) There is a (small) progress measure
$\mu : V \to \mathbb{J}_{n,d/2}$

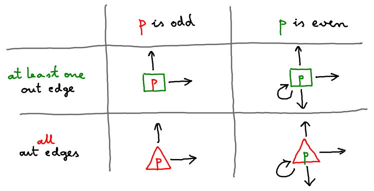
(2) There is a succinct progress measure
$\kappa : V \to \mathbb{L}_{n,d/2}$

ORDERED TREE CODING



THE PARITY KEYBOARD DECOMPOSITION





A PROGRESS MEASURE

$\boxed{\mu : V \to \mathbb{N}^{d/2}}$

- Edge (v,u) is progressive if
  - $\mu(v)|_{\pi(v)} \geq_{lex} \mu(u)|_{\pi(v)}$ and $\pi(v)$ is even
  - $\mu(v)|_{\pi(v)} >_{lex} \mu(u)|_{\pi(v)}$ and $\pi(v)$ is odd
- Vertex v is progressive if
  - $v \in V_{Even}$ and at least one outedge is progressive
  - $v \in V_{Odd}$ and all outedges are progressive

$\boxed{\mu \text{ is a progress measure if all vertices are progressive}}$

THE SMALL PROGRESS MEASURE THEOREM

Thm TFAE

(1) There is a parity keyboard decomposition of V

(2) There is a small progress measure
$\mu : V \to \underbrace{\boxed{\{0,1,2,\ldots,n\}^{d/2}}}_{\mathbb{J}_{n,d/2}}$

(3) There is a positional □-dominion strategy on V

THE LIFTING ALGORITHM

1. Let $\mu(v) := (0,0,\ldots,0) \in \mathbb{J}^{\top}_{n,d/2}$ for all $v \in V$
2. While there is a vertex v that is not progressive
   do let $\mu := \mathrm{MinLift}_v(\mu)$
3. Return
   (a) the progressive vertices (winning positions)
   (b) the progressive edges (positional winning strategy)

COMPLEXITY OF THE LIFTING ALGORITHM

- Time: $O\left(\sum_{v \in V} d \deg(v) \left|\mathbb{J}_{n,d/2}\right|\right) = O\left(dm \cdot \left|\mathbb{J}_{n,d/2}\right|\right)$
  $= n^{\frac{d}{2}+O(1)}$
- Space: $O(dn)$

SUCCINT ADAPTIVE MULTI-COUNTERS

$\mathbb{B} = \{0, 1\}$

$$\boxed{\mathbb{L}_{n,d/2} = \left\{(s_{d-1}, s_{d-3}, \ldots, s_1) : s_i \in \mathbb{B}^* \text{ and } \sum_{i=1}^{d/2} |s_{2i-1}| \leq \lceil \lg n \rceil \right\}}$$

Fact $\left|\mathbb{L}_{n,d/2}\right| \leq 2^{\lceil \lg n \rceil \cdot (1 + \lceil \lg \frac{d}{2} \rceil)} = n^{\lg d + O(1)}$

THE ~~~~ succinct LIFTING ALGORITHM

1. Let $\mu(v) := \cancel{(0,0,\ldots,0) \in \mathbb{J}^{\top}_{n,d/2}}$ $\bot \in \mathbb{L}_{n,d/2}$ for all $v \in V$
2. While there is a vertex v that is not progressive
   do let $\mu := \cancel{\mathrm{MinLift}}_v(\mu)$ MinLift in $\mathcal{L}$
3. Return
   (a) the progressive vertices (winning positions)
   (b) the progressive edges (positional winning strategy)

COMPLEXITY OF THE SUCCINCT LIFTING ALGORITHM

- Time: $O\left(\sum_{v \in V} \overset{\log d \cdot \log n}{\cancel{d}} \cdot \deg(v) \cdot \left|\overset{\mathbb{L}_{n,d/2}}{\mathbb{J}_{n,d/2}}\right|\right) = \tilde{O}\left(m \cdot \left|\mathbb{L}_{n,d/2}\right|\right)$
  $= \boxed{n^{\log d + O(1)}}$
- Space: $O(\overset{\log d \cdot \log n}{\cancel{d}}\, n) = \boxed{\tilde{O}(n)}$

THE SIZE OF $\mathbb{L}_{n,d/2}$

- $\left|\mathbb{L}_{n,d/2}\right| \leq 2^{\lceil \lg n \rceil} \cdot \binom{\lceil \lg n \rceil + \frac{d}{2}}{\frac{d}{2}}$
- $$\left|\mathbb{L}_{n,d/2}\right| = \begin{cases} O\left(n \cdot \lg^{d/2} n\right) & \text{if } d = O(1) \\ O\left(n^{1+o(1)}\right) & \text{if } d = o(\log n) \\ \tilde{\Theta}\left(n^{\lg(\delta+1) + \lg(e_\delta) + 1}\right) & \text{if } d = 2\lceil \delta \lg n \rceil \\ O\left(d \cdot n^{\lg\left(\frac{d}{\lg n}\right) + 1.45}\right) & \text{if } d = \omega(\log n) \end{cases}$$
  where $e_\delta = \left(1 + \frac{1}{\delta}\right)^{\delta}$

RUNNING TIMES

d	CJKLS'17	JL'17	GI-J'17	FJSSW'17
$O(1)$		$O\left(m \cdot \eta \cdot \lg^{\frac{d}{2}+1} \eta\right)$		$O\left(m\eta \cdot \lg^{d-1} \eta\right)$
$o(\log \eta)$		$O\left(m \cdot \eta^{1+o(1)}\right)$		
$2\lceil \delta \cdot \lg \eta \rceil$		$\tilde{O}\left(m \cdot \eta^{\lg(\delta+1) + \lg(e_\delta) + 1}\right)$		
$\lceil \lg \eta \rceil$	$O(n^5)$	$O\left(m \cdot \eta^{2.38}\right)$	$O\left(m \cdot n^{2.55}\right)$	
$\omega(\log \eta)$	$O\left(n^{\lg d + 6}\right)$	$O\left(d \cdot m \cdot \eta^{\lg\left(\frac{d}{\lg \eta}\right) + 1.45}\right)$		
$\Omega(\log^2 \eta)$			$O\left(d \cdot m \cdot n^{\lg\left(\frac{d}{\lg n}\right) + 1.45}\right)$	$O\left(d\, m\, \eta^{\lg\left(\frac{d}{\lg \eta}\right) + 1.45}\right)$