Dynamics of Distributed Updating in Fisher Markets

Generalized / Damped Proportional Response Dynamics (PRD)

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TAKE HOME MESSAGE

Market Dynamics to Reach Equilibrium

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Optimization Iterative Processes: **Gradient / Mirror Descent**

In Fisher CES Markets,

Tâtonnement Price Updates by sellers ⇔ **Gradient Descent** [C., Cole, Devanur, STOC 2013]

Proportional Response Spending Updates by buyers ⇔ **Mirror Descent** [this work, ACM EC 2018]

Independent Interest from Optimization:
We present new Mirror Descent analyses to handle new and broad classes of **(Strongly) Bregman Convex or Convex-Concave Functions**.

MOTIVATION

A major goal in Algorithmic Game Theory: **justify equilibrium concepts (often defined statically) from an algorithmic and complexity perspective**.

How? Give **efficient natural algorithm/dynamics** which can be run in the **highly distributed market environment**, and converge to market equilibrium.

(The following centralized algorithm techniques won't suffice: Ellipsoid Method, Interior Point Method, Flow-Based Combinatorial Algorithms.)

MARKET DYNAMICS

Buyer i has budget e_i, and CES utility function ($\rho_i \in [-\infty, 1]$)

$$u_i(\mathbf{x}_i) = (a_{i1}(x_{i1})^{\rho_i} + \cdots + a_{in}(x_{in})^{\rho_i})^{1/\rho_i}$$

For Substitute Range ($\rho_i \geq 0$) [Wu, Zhang STOC 2007; Zhang ICALP 2009],

$$\text{spending}_{ij}(t+1) \equiv b_{ij}(t+1) \leftarrow e_i \cdot \frac{a_{ij}\cdot (x_{ij}(t))^{\rho_i}}{\sum_k a_{ik}\cdot (x_{ik}(t))^{\rho_i}}.$$

We spotted that the above rule is equivalent to mirror descent on the convex function

$$-\sum_{i,j} \frac{b_{ij}}{\rho_i} \log \frac{a_{ij}(b_{ij})^{\rho_i - 1}}{(\sum_h b_{hj})^{\rho_i}} \quad \text{subject to } \forall i, \ \sum_j b_{ij} = 1.$$

For Complementary Range ($\rho_i \leq 0$), the function becomes concave, and the mirror ascent rule is our **generalized PRD**:

$$b_{ij}(t+1) \leftarrow e_i \cdot \frac{a'_{ij}\cdot (\text{total spending on good } j \text{ at time } t)^{\rho_i/(\rho_i-1)}}{\sum_k a'_{ik}\cdot (\text{total spending on good } k \text{ at time } t)^{\rho_i/(\rho_i-1)}}$$

For Mixed Range, similar (but a bit more involved) update rule converges. (Bregman Convex-Concave Functions appear.)

THEOREM. Convergence to market equilibrium for **full range of CES utility functions**.

OPTIMIZATION

$(\sigma_X, \sigma_Y, L_X, L_Y)$**-Strongly Bregman Convex-Concave Function:**

$$\begin{aligned} -L_Y \cdot d_h(\mathbf{y}', \mathbf{y}) + \sigma_X \cdot d_g(\mathbf{x}', \mathbf{x}) &\leq f(\mathbf{x}', \mathbf{y}') - \text{linearization at } f(\mathbf{x}, \mathbf{y}) \\ &\leq -\sigma_Y \cdot d_h(\mathbf{y}', \mathbf{y}) + L_X \cdot d_g(\mathbf{x}', \mathbf{x}). \end{aligned}$$

d_g, d_h are ANY Bregman divergences.

$$\mathbf{x}^{t+1} \leftarrow \arg\min_{\mathbf{x}} \{ \langle \nabla_{\mathbf{x}} f(\mathbf{x}^t, \mathbf{y}^t) \ , \ \mathbf{x} - \mathbf{x}^t \rangle + 2L_X \cdot d_g(\mathbf{x}, \mathbf{x}^t) \}$$

$$\mathbf{y}^{t+1} \leftarrow \arg\min_{\mathbf{y}} \{ - \langle \nabla_{\mathbf{y}} f(\mathbf{x}^t, \mathbf{y}^t) \ , \ \mathbf{y} - \mathbf{y}^t \rangle + 2L_Y \cdot d_h(\mathbf{y}, \mathbf{y}^t) \}$$

THEOREM. If σ_X, σ_Y are strictly positive, linear pointwise convergence toward the saddle point.

THEOREM. If $\sigma_X, \sigma_Y \geq 0$, $\mathcal{O}(1/T)$ empirical convergence toward the saddle point.

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