Approximate Graph Embeddings in the Cloud

Matthias Rost
Internet Network Architectures
Technical University of Berlin

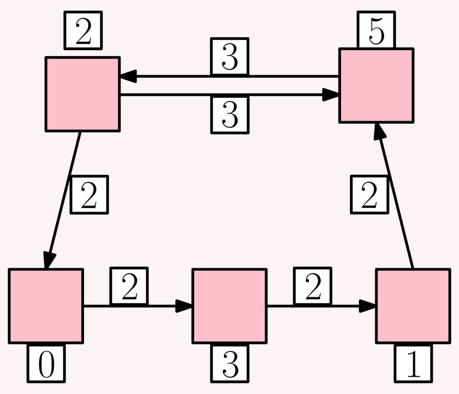
Stefan Schmid
Communication Technologies
University of Vienna

The Virtual Network Embedding Problem

Substrate $G_S = (V_S, E_S)$

Represents **physical network**

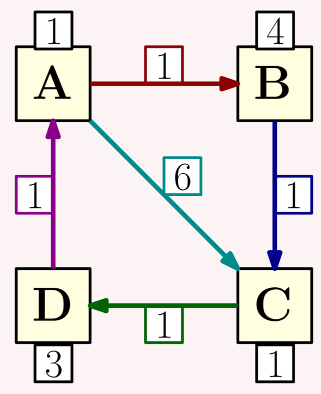
- capacities $d_S : G_S \to \mathbb{R}_{\geq 0}$



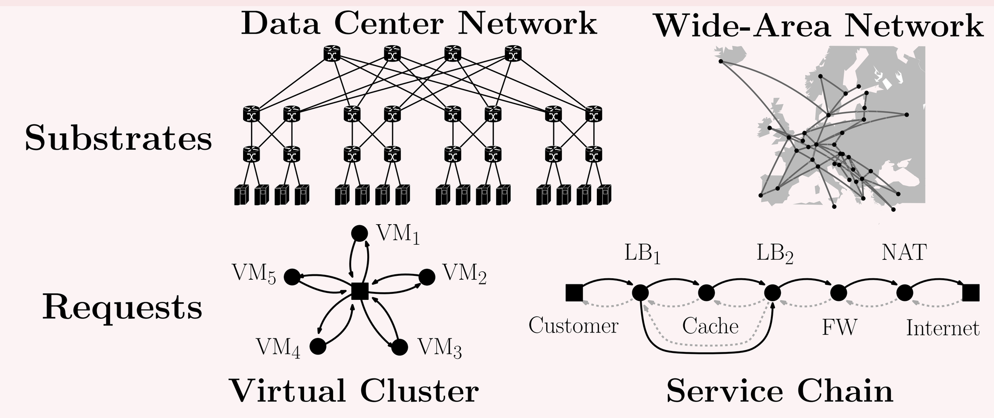
Virtual Network Request $G_r = (V_r, E_r)$

Represents **workload**

- demands $d_r : G_r \to \mathbb{R}_{\geq 0}$
- profit $p_r \in \mathbb{R}_{\geq 0}$
- mapping restrictions
  - $V_S^i \subseteq V_S$ for $i \in V_r$
  - $E_S^{i,j} \subseteq E_S$ for $(i,j) \in E_r$



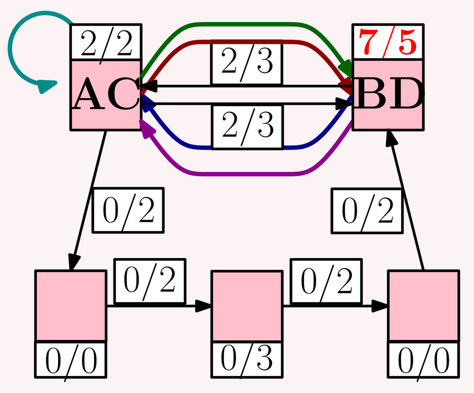
Example Substrates & Requests



Valid Mappings $m_r \in \mathcal{M}_r$ and Feasible Embeddings

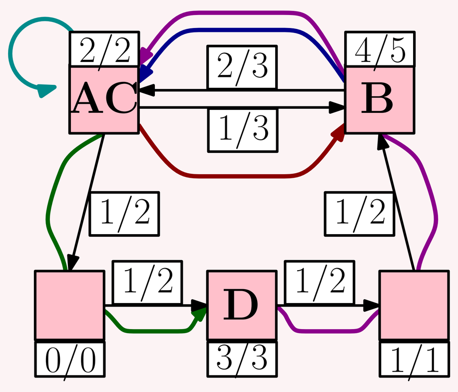
A valid mapping $m_r = (m_V, m_E) \in \mathcal{M}_r$, with $m_V : V_r \to V_S$ and $m_E : E_r \to \mathcal{P}(E_S)$, satisfies . . .

valid connectivity: $m_V(i) \overset{m_E(i,j)}{\rightsquigarrow} m_V(j)$
valid node mapping: $m_V(i) \in V_S^i$
valid edge mapping: $m_E(i,j) \subseteq E_S^{i,j}$



Valid Mapping

A feasible embedding is valid and respects capacity constraints



Feasible Embedding

Problem Statement

Given: substrate G_S and a set of requests $\{G_r\}_{r \in \mathcal{R}}$
Task: embed subset $\mathcal{R}' \subseteq \mathcal{R}$ *feasibly* maximizing $\sum_{r \in \mathcal{R}'} p_r$

Previous Works

- survey from 2013 lists more than 100 papers [1]
- **only heuristics & exact algorithms known**

Randomized Rounding Framework

Enumerative Packing Formulation

$$\max \sum_{r \in \mathcal{R}} \sum_{m_r^k \in \mathcal{M}_r} f_r^k \cdot p_r$$
$$\sum_{m_r^k \in \mathcal{M}_r} f_r^k \leq 1 \qquad \forall r \in \mathcal{R}$$
$$\sum_{r \in \mathcal{R}} \sum_{m_r^k \in \mathcal{M}_r} f_r^k \cdot A(m_r^k, x) \leq d_S(x) \qquad \forall x \in R_S$$
$$f_r^k \in [0,1] \qquad \forall r \in \mathcal{R}, m_r^k \in \mathcal{M}_r$$

result are 'convex' combinations of mappings
$\mathcal{D}_r = \{(f_r^k, m_r^k) | f_r^k \geq 0, m_r^k \in \mathcal{M}_r\}$ for $r \in \mathcal{R}$

Classic Randomized Rounding à la Raghavan & Thomspon [2]

Algorithm: VNEP Approximation
`// perform preprocessing`
compute *optimal* LP solution
compute $\{\mathcal{D}_r\}_{r \in \mathcal{R}}$ from LP solution
do
  solution $\leftarrow$ ROUNDING($\{\mathcal{D}_r\}_{r \in \mathcal{R}}$)
while (*solution not* (α, β, γ)*-approximate and rounding tries not exceeded*)

Algorithm 2: ROUNDING
Input : Decomp. LP Solution $\{\mathcal{D}_r\}_{r \in \mathcal{R}}$
foreach $r \in \mathcal{R}$ **do**
  choose m_r^k **with probability** f_r^k
end
return *solution*

Derived Heuristics

Vanilla Rounding rounds n many solutions and return the solution maximizing the profit ($\mathrm{RR}_{\mathrm{MaxProfit}}$) or the one minimizing the resource augmentations ($\mathrm{RR}_{\mathrm{MinLoad}}$).

Heuristic Rounding discards rounded mappings whose inclusion would exceed capacities $\to$ no resource augmenations.

Our Randomized Rounding Algorithm returns (α, β, γ)-approximate solutions for the VNEP *with high probability*. [3], [4]

$$\underbrace{\alpha = 1/3}_{\text{profit}} \qquad \underbrace{\beta = 1 + \varepsilon \cdot \left(2 \cdot \underbrace{\Delta(V_S)}_{\leq |\mathcal{R}| \cdot \max_{r \in \mathcal{R}} |V_r|} \cdot \log(|V_S|)\right)^{1/2}}_{\text{node resource augmentations}} \qquad \underbrace{\gamma = 1 + \varepsilon \cdot \left(2 \cdot \underbrace{\Delta(E_S)}_{\leq |\mathcal{R}| \cdot \max_{r \in \mathcal{R}} |E_r|} \cdot \log(|E_S|)\right)^{1/2}}_{\text{edge resource augmentations}}$$

$$\underbrace{\varepsilon = \max_{r \in \mathcal{R}, x \in R_S} d_{max}(r,x)/d_S(x) \leq 1}_{\text{max demand-to-capacity ratio}} \qquad \Delta(X) = \max_{x \in X} \sum_{r \in \mathcal{R}} \left(\underbrace{A_{max}(r,x)}_{\substack{\text{max alloc. on resource } x \\ \text{by any valid mapping of } r}} \Big/ \underbrace{d_{max}(r,x)}_{\substack{\text{max single demand for} \\ \text{resource } x \text{ by request } r}} \right)^2$$

Main Contribution: Decomposable Linear Programming Formulations

Classic Multi-Commodity Flow LP

$\mathbf{y^u_{r,i}} \in [0,1]$: maps node $i \in V_r$ on V_S
$\mathbf{z^{u,v}_{r,i,j}} \in [0,1]$: maps edge $(i,j) \in E_r$ on $(u,v) \in E_S$

$$\sum_{(u,v) \in \delta^+(u)} \mathbf{z^{u,v}_{r,i,j}} - \sum_{(v,u) \in \delta^-(u)} \mathbf{z^{v,u}_{r,i,j}} = \mathbf{y^u_{r,i}} - \mathbf{y^u_{r,j}}$$

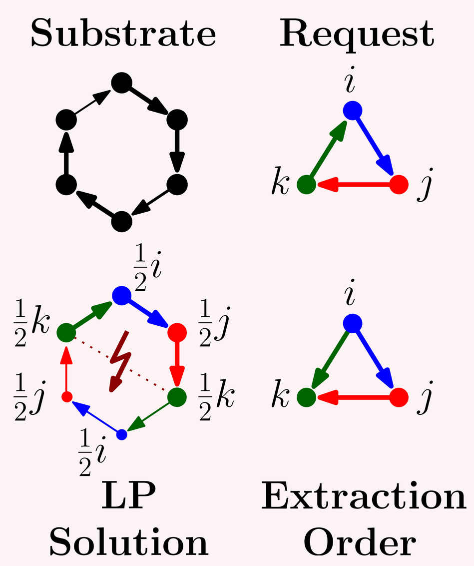
Local Connectivity Property

Given a (fractional) mapping of $i \in V_r$ to $u \in V_S$, a 'valid' mapping can be recovered for edges incident to i and their respective endpoints.

Consider $(i,j) \in E_r$ and $y^u_{r,i} > 0$, then $\exists P : u \rightsquigarrow v$ with $y^v_{r,j} > 0$ and $z^e_{r,i,j} > 0$ for $e \in P$.

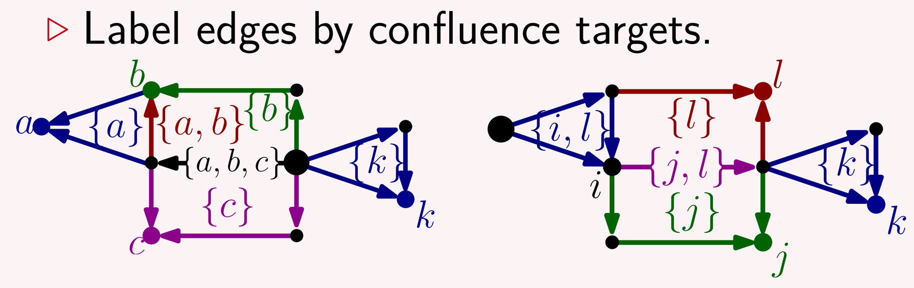
Consider $(k,i) \in E_r$ and $y^u_{r,i} > 0$, then $\exists P : w \rightsquigarrow u$ with $y^w_{r,k} > 0$ and $z^e_{r,k,i} > 0$ for $e \in P$.

Thm: MCF LP solutions cannot be decomposed into valid mappings.

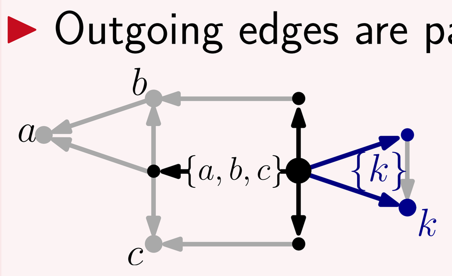


Idea of Novel LP Formulation [4]

- $\{ \langle (i,j), (j,k) \rangle, \langle (i,k) \rangle \}$ is **confluence**.
- **Confl. targets need to be decided a priori.**
  - Label edges by confluence targets.



- Outgoing edges are partitioned into bags:



- Edges with overlapping labels are placed into same bag.

- **Extraction Width (of specific order):** maximal edge bag size plus one.

Extraction Width & Formulation Size [4]

- **For spec. extraction order $\mathrm{G}_r^{\mathcal{X}}$, the size of our novel LP formulation is $\mathcal{O}(|\mathrm{G_S}|^{\mathrm{ew}_{\mathcal{X}}(\mathrm{G}_r^{\mathcal{X}})} \cdot |\mathrm{G_r}|)$.**
- **Thm: Finding order of min width is $\mathcal{NP}$-hard.**

(FPT-)Approximating the VNEP [3], [4]

- **LP Solutions can be decomposed into set $\mathcal{D}_r$ of weighted mappings (cf. Packing LP).**
- **FPT-approximations for the VNEP; poly.-time approximations for cactus request graphs.**

Hardness of Approximating the VNEP [5]

We cannot do better than FPT: computing valid mappings is $\mathcal{NP}$-complete for planar graphs.

Computational Evaluation [3]

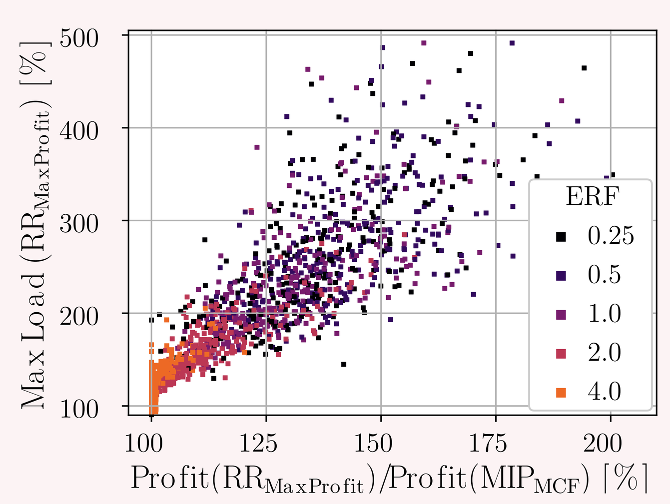
Requests
- Cactus Graphs

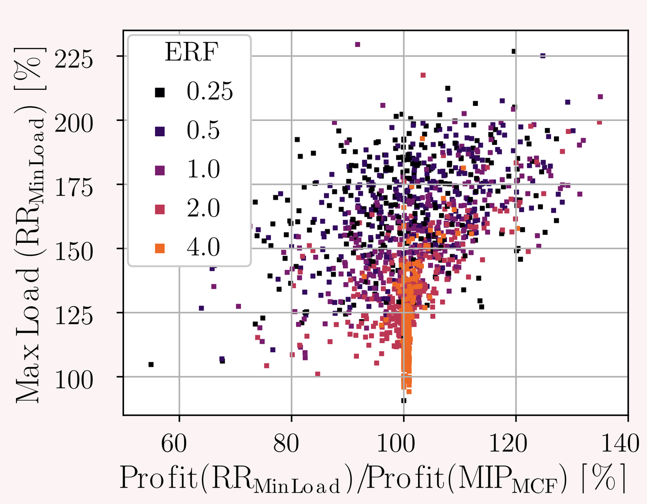
Substrate
- GEANT Network

Parameter Space
- Node Resources
- Edge Resources
- Number Requests

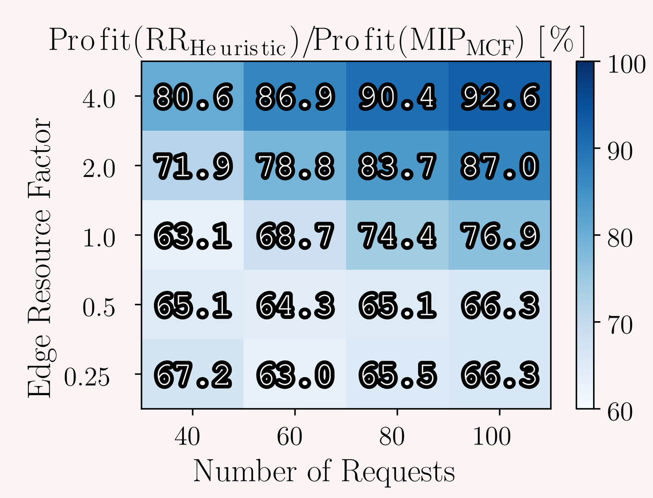
Baseline
- Integer Program

Vanilla Rounding: Max Profit & Min Augmentations





Heuristic Rounding



References

[1] A. Fischer, J. F. Botero, M. T. Beck, H. De Meer, and X. Hesselbach, "Virtual network embedding: a survey", *IEEE Communications Surveys & Tutorials*, vol. 15, no. 4, 2013.

[2] P. Raghavan and C. D. Thompson, "Provably Good Routing in Graphs: Regular Arrays", in *Proc. 17th ACM STOC*, 1985.

[3] M. Rost and S. Schmid, "Virtual Network Embedding Approximations: Leveraging Randomized Rounding", in *Proc. IFIP Networking*, 2018.

[4] ——, "(FPT-)Approximation Algorithms for the Virtual Network Embedding Problem", Tech. Rep., Mar. 2018, (under submission). [Online]. Available: http://arxiv.org/abs/1803.04452.

[5] ——, "Charting the Complexity Landscape of Virtual Network Embeddings", in *Proc. IFIP Networking*, 2018.

Contact: Matthias Rost - mrost@inet.tu-berlin.de