Subquadratic algorithms for the diameter and the sum of pairwise distances in planar graphs

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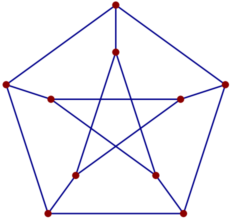
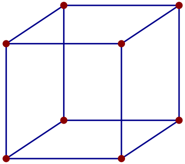
Context

Basic parameters of a graph
(edge-weighted)

- number of vertices
- number of edges
- average/max/min degree
- diameter
- average distance / sum of distances
- girth
- connectivity
- ...





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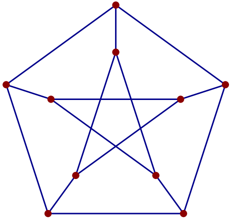

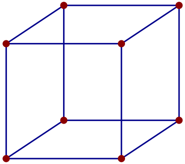

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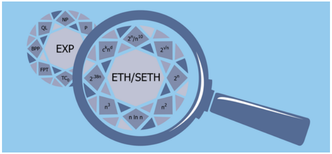
Fine-grained complexity
Conditional lower bounds
based on a few *basic* assumptions



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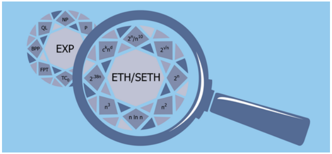

[Roditty, Vassilevska Williams]

Computing the diameter of (unweighted) graphs with n vertices and $O(n)$ edges in $O(n^{2-\delta_0})$ time

$$\Downarrow$$

SAT with m variables can be solved in $O\left((2-\delta_1)^m\right)$ time

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Strong Exponential Time Hypothesis (SETH) is false

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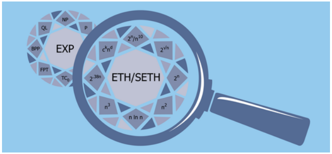

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Strong Exponential Time Hypothesis (SETH) is false

- Same result holds for *sum of distances*
- Find graph classes where the bound does not hold

Context

- Find graph classes where we can compute the diameter and the sum of distances in $O(n^{2-\delta_0})$ time
- Treewidth k in $O(n \operatorname{polylog} n)$ time
  [C., Knauer; Abboud, Vassilevska Williams, Wang]

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Our result:

Diameter and sum of the pairwise distances in **planar** graphs with positive edge weights in $O(n^{11/6} \operatorname{polylog} n)$ expected time

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- Best previous result for planar graphs: $O(n^2(\log\log n)^4/\log n)$

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  [Wulff-Nilsen]
- Best next result for planar graphs: $\tilde{O}(n^{10/6})$

  [Gawrychowski, Kaplan, Mozes, Sharir, Weimann]

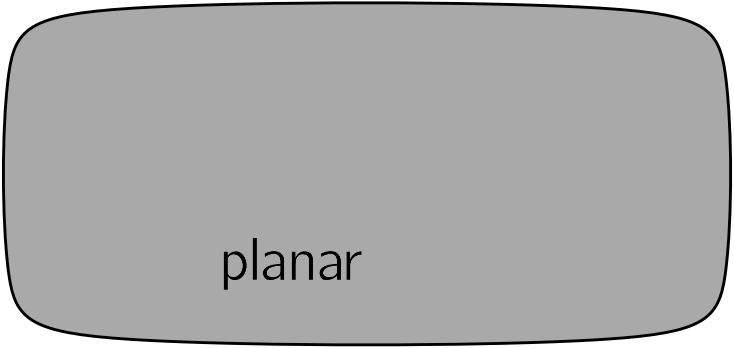
How?

Graph Algorithms & **Computational Geometry**

- r-division of planar graphs (includes separators)
- preprocessing of planar graphs for sum/max weight inside query subgraph
- dense-distance graphs (includes multiple-source shortest paths)
- (additively-weighted) Voronoi diagrams in planar/plane graphs
- abstract Voronoi diagrams
- randomized incremental constructions
- ...

Using tools by several researchers as black box.

Using r-divisions



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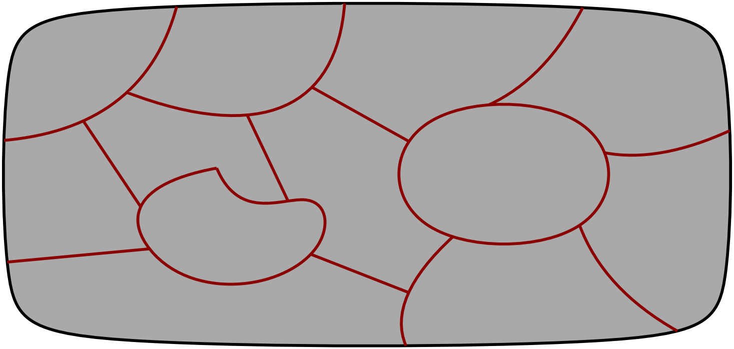
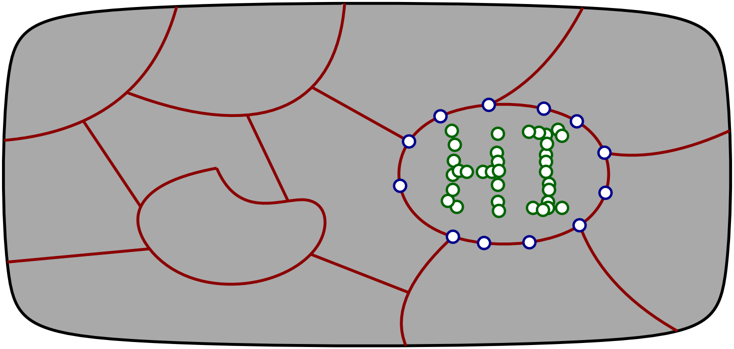


$\Theta(n/r)$ pieces, $\Theta(r)$ vertex per piece

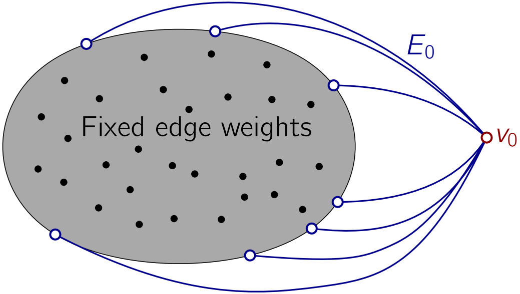
Using r-divisions



$\Theta(n/r)$ pieces, $\Theta(r)$ vertex per piece
$O(\sqrt{r})$ boundary vertices per piece

Main new ingredient

Data: planar graph H with distinguished vertex $v_0 \in V(H)$.
Weights for $E(H) \setminus E_0$ fixed, where E_0 edges of H incident to v_0.



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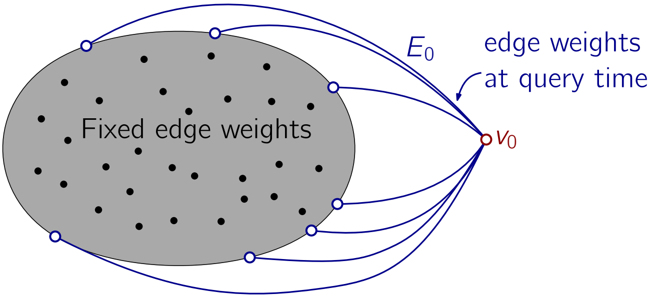

Query: given the weights for the edges E_0 at query time,
report $\sum_{u \in V(H)} d_H(v_0, u)$ and $\max_{u \in V(H)} d_H(v_0, u)$

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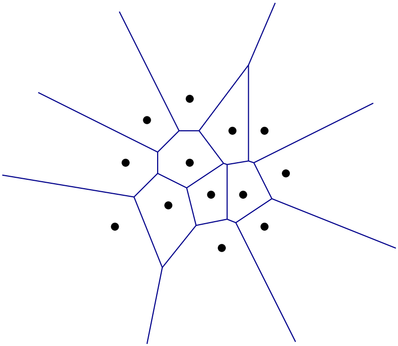
Preprocessing time: $O(b^3 r^2)$ time, where $r = |V(H)|$ and $b = |E_0|$
Query time: $O(b \operatorname{polylog} b)$ expected time
Using: Voronoi diagrams, randomized incremental construction

Preprocessing $O(r^{10})$ would also lead to subquadratic time.

Voronoi diagrams – Classical



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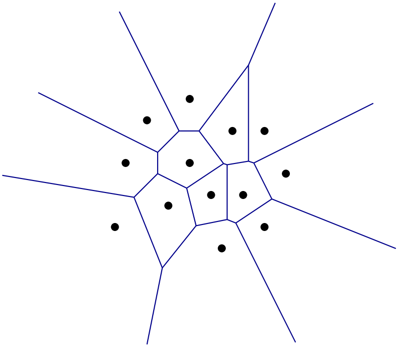
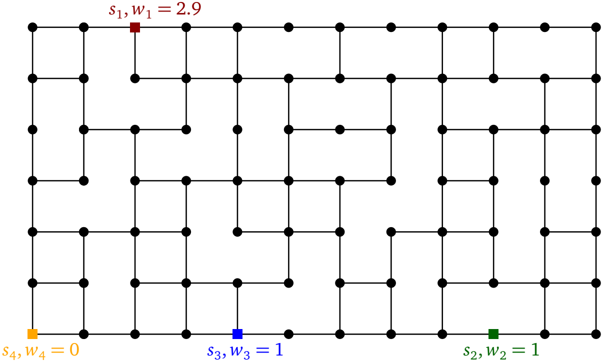


- Can be generalized to arbitrary metric spaces
- Additive-weights on the sites
- Abstract Voronoi diagrams: just needs “bisectors” in the plane and **some** other properties

Additively-weighted VD in plane graphs



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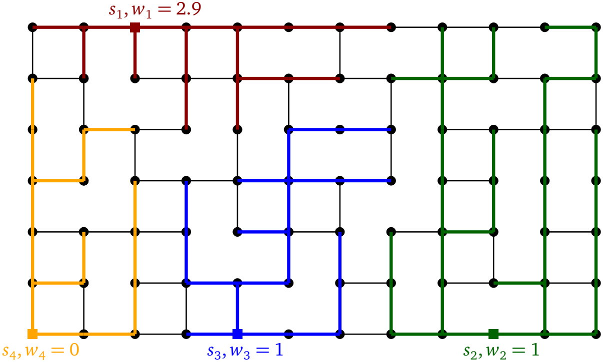

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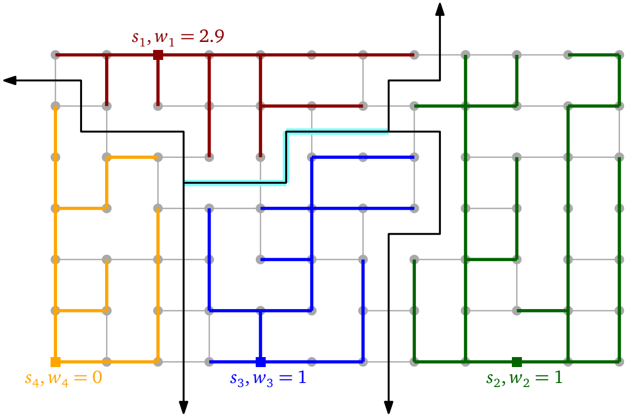


Bisectors are curves defined by the dual graph
Edges of the VD described by 4 sites

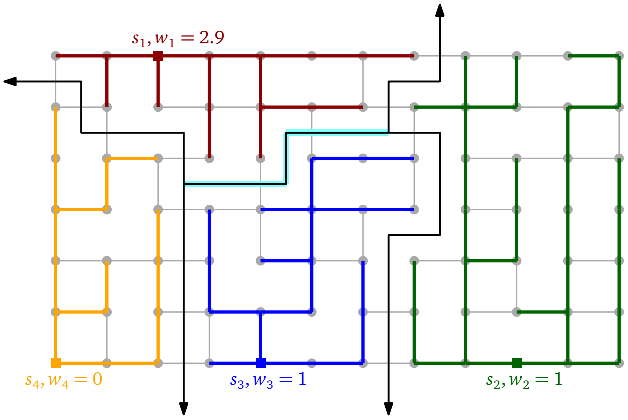
Additively-weighted VD in plane graphs

Via [Klein, Mehlhorn, Meiser '93] (randomized incremental construction):

- we can "compute" the additively-weighted VD in a plane graph
- all sites must be co-facial
- expected $O(s \log s)$ elementary operations, where s nb of sites
- computing the VD of 4 sites is an **elementary operation**
- the output is a combinatorial description of the VD in the dual graph

Independent of the size of the graph!!

Description of the output



How to use this?

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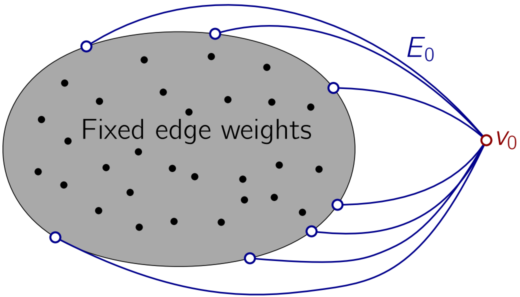

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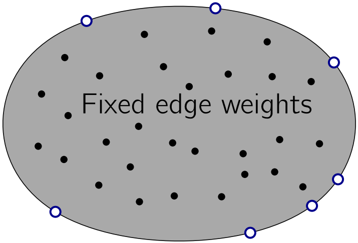

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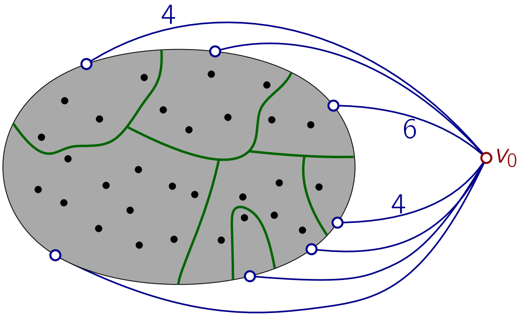

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Query time: weights at E_0 $\Rightarrow$ weights at sites
in $O(b \log b)$ time we get the Voronoi regions
have to "collect" the data from each region . . . preprocessing

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- Sum of distances and diameter for H-minor-free graphs
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Many THANKS for your time!