Approximate Modularity Revisited

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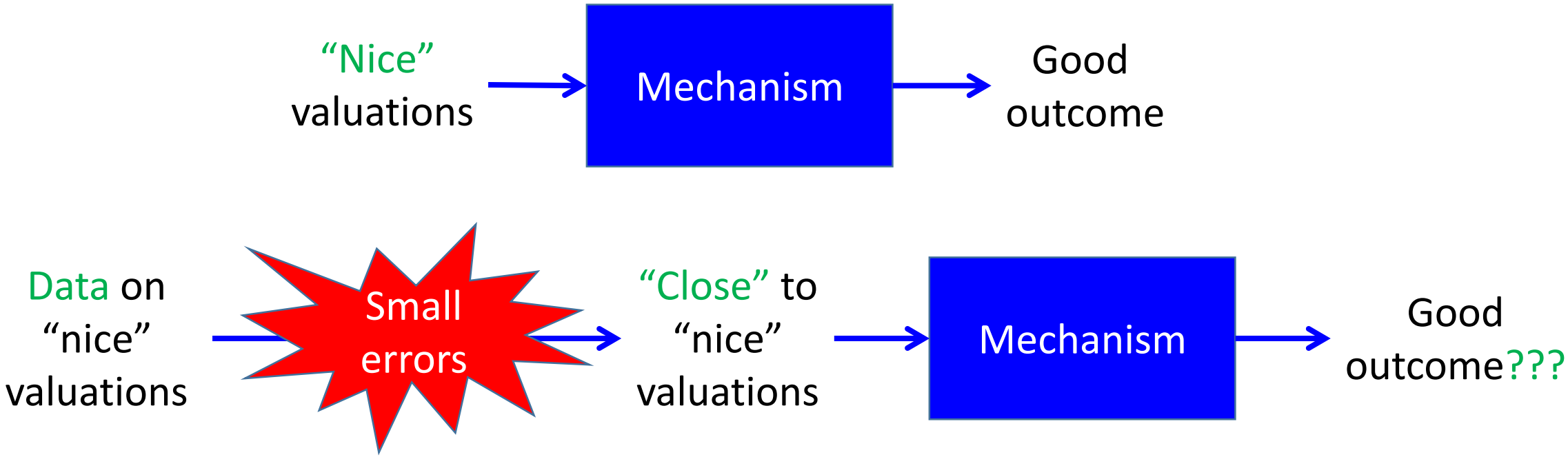
Joint work with Uriel Feige, Michal Feldman

STOC'17, HALG'18

*Part of this work was done at MSR Herzliya

Motivation: Robustness/stability

Mechanism and market design as a "data-driven enterprise".



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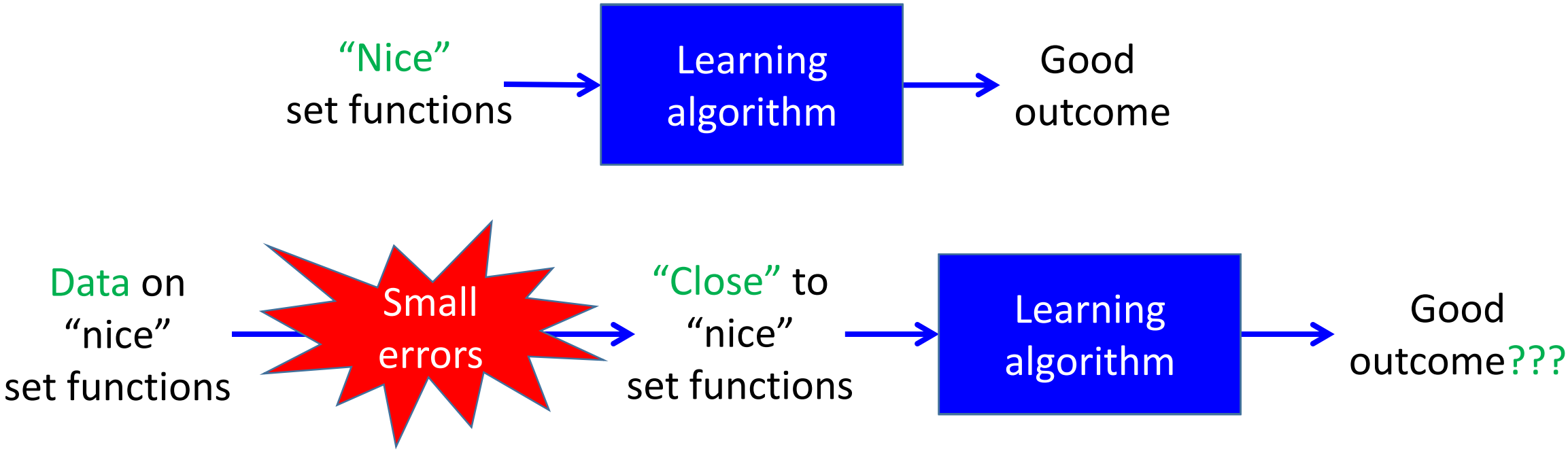


High-level research agenda

What happens when set functions are only "close" to being "nice"?

Many sub-questions:

- Notions of closeness.
- Optimization – can good approximation ratios still be achieved?
- Mechanism design – do good economic properties continue to hold?
- Learning – can nice set functions be recovered?

- One take-away: Basic questions still open; interesting math involved

Set functions

Universe of n items.

Function $f: \{0,1\}^n \to \mathbb{R}$ assigns values $f(S)$ to sets of items.

Examples:

- Items for sale in a combinatorial auction.
  Value of set: willingness of a bidder to pay for bundle of items.
- Items are vertices in an edge-weighted graph.
  Value of set: total weight of edges in induced cut.

Representation of set functions

Explicit: 2^n entries.

Value queries: upon query S learn $f(S)$.

- In a combinatorial auction, one may ask a bidder how much she is willing to pay for the bundle.
- Given a set S of vertices, one can compute in polytime the total weight of edges in the cut $(S, \bar{S})$.

Other types of queries (such as demand queries) have been studied.

Classes of "nice" set functions

Some classes of nice set functions with polynomial representations:

Additive $f(S) = \sum_{i \in S} f(i)$, and $f(\emptyset) = 0$.

Linear $f(S) = f(\emptyset) + \sum_{i \in S} f(i)$.

Capped-additive $f(S) = \min\{1, g(S)\}$ where g is additive.

Nice properties but not necessarily a polynomial representation:

Submodular $f(S) + f(T) \geq f(S \cap T) + f(S \cup T)$.

Subadditive $f(S) + f(T) \geq f(S \cup T)$, and $f(\emptyset) = 0$.

Optimization problems for “nice” set functions

Linear set functions:

- Many optimization problems are easy.

Submodular set functions (cuts, valuations with diminishing marginals):

- Maximization subject to constraints (e.g., a cardinality constraint) can be solved approximately.

General set functions:

- Many optimization problems are computationally hard to approximate.

Motivations for “close to” nice set functions

- Answers to value queries might not be exact due to noise in measurements.
- Rounding errors.
- Computing approximate values may be cheaper than computing exact values.
- The functions might not be exactly nice (e.g., valuation functions of bidders need not be exactly submodular).

Some related work

Functions on continuous domains (rather than discrete hypercube):

- Hyers [1941], …

Data-driven/sample-based optimization:

- Bertsimas and Thiele [2014], Singer and Vondrak [2015], Hassidim and Singer [2016], Balkanski, Rubinstein, and Singer [2016], …

Approximate submodularity/convexity/substitutes:

- Lehmann, Lehmann, and Nisan [2006], Das and Kempe [2011], Belloni, Liang, Narayanan, and Rakhlin [2015], Roughgarden, T.C., and Vondrak [2016], …

Learning submodular functions:

- Balcan and Harvey [2011], …

…

Our Results

We follow-up on Chierichetti et al. [FOCS'15]

They suggested to start with an "easy" case: a function f close to linear.

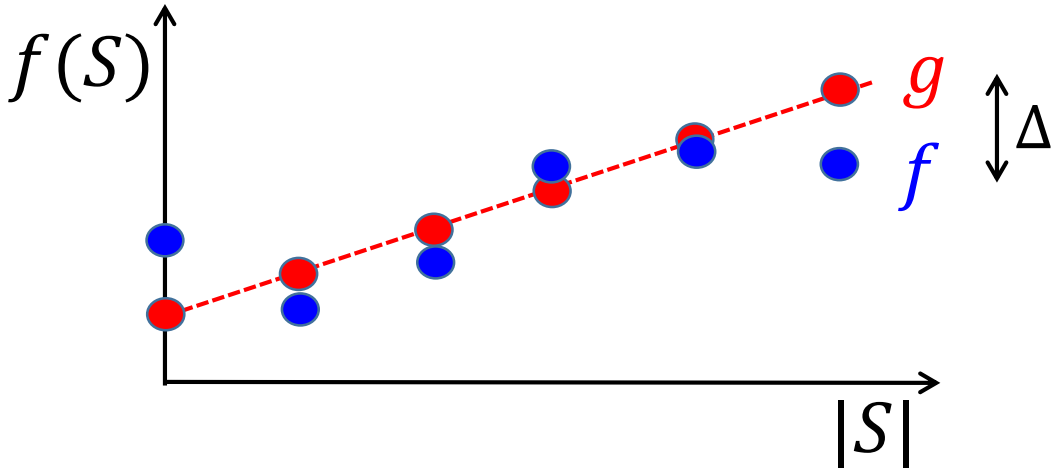
Posed 2 questions:

- To what extent are different measures of closeness to linearity related to each other?
  - being pointwise close to a linear function,
  - nearly satisfying properties of linear functions.
- How to learn a linear function h that is close to f?

Definition: Δ-linear set functions

Linear set function: $g(S) = g(\emptyset) + \sum_{i \in S} g(i)$.

Given $\Delta \geq 0$, a set function f is Δ-linear if there is a linear set function g such that $|f(S) - g(S)| \leq \Delta$ pointwise for every set S.



Δ-linear: pointwise close to linear up to $\pm\Delta$

Definition: ε-modular set functions

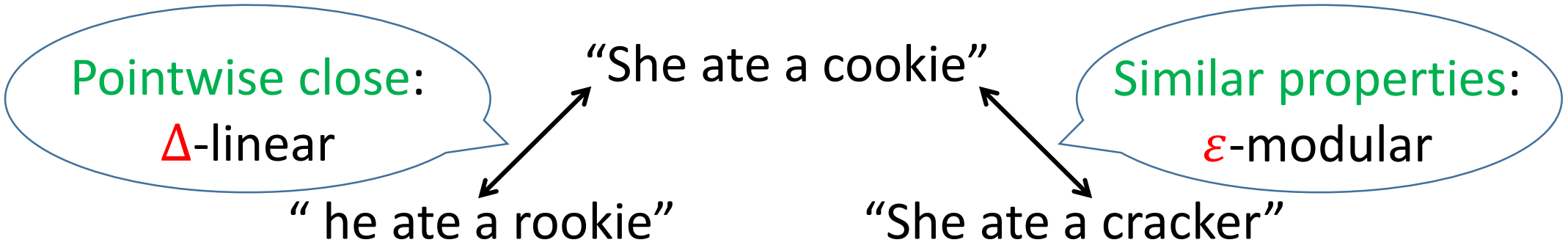
A set function is linear if and only if it is modular:

$f(S) + f(T) = f(S \cap T) + f(S \cup T)$ for all sets S and T.

A set function is ε-modular if

$|f(S) + f(T) - f(S \cap T) - f(S \cup T)| \leq \varepsilon$ for all sets S and T.

“Syntactic” versus “semantic” closeness



Δ-linear: pointwise close to linear up to $\pm\Delta$

Relating the closeness notions

Recall a set function is ε-modular if

$|f(S) + f(T) - f(S \cap T) - f(S \cup T)| \leq \varepsilon$ for all sets S and T.

Every Δ-linear function is ε-modular for $\varepsilon \leq 4\Delta$.

Is it true that every ε-modular function is Δ-linear for $\Delta \leq O(\varepsilon)$?

Δ-linear: pointwise close to linear up to $\pm\Delta$
ε-modular: satisfies modularity eqs. up to $\pm\epsilon$

Results for ε-modularity

Assaf Naor directed us to previous work from a different community [Kalton and Roberts; 1983] showing ε-modular is Δ-linear for:

- $\Delta \leq 44.5\varepsilon$

We adapt techniques from [KR'83] and [Chierichetti+'15] to show:

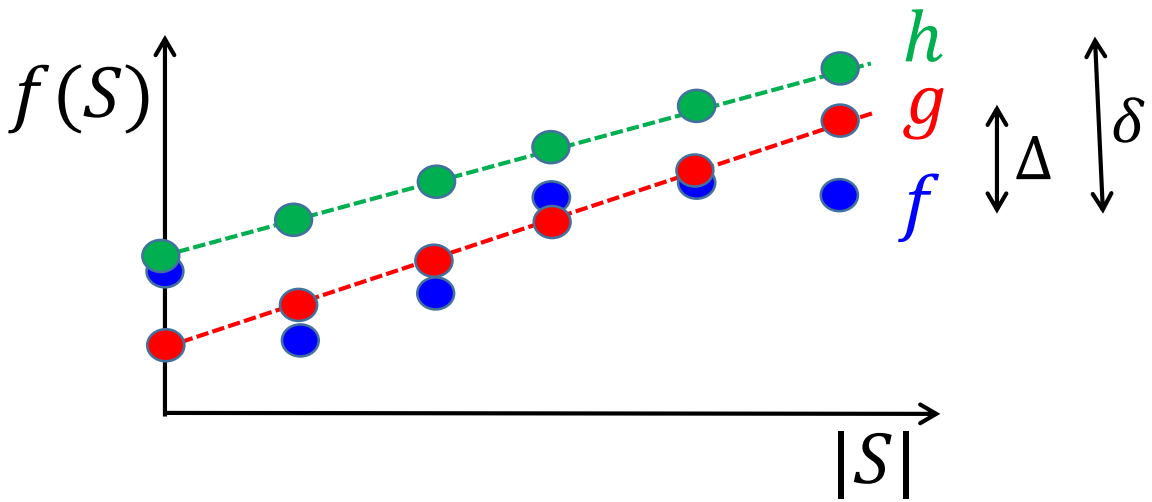
- $\Delta < 13\varepsilon$, with better bounds for special classes of set functions.
  - E.g., $\Delta \leq \frac{1}{2}\varepsilon$ for symmetric functions.
- $\Delta \geq \varepsilon$ for some set function (with $n = 70$).

Δ-linear: pointwise close to linear up to $\pm\Delta$

Learning Δ-linear set functions

Suppose we are given value query access to a Δ-linear function f.
Using polynomially many value queries, output a linear function h satisfying $|f(S) - h(S)| \leq \delta$ for every set S.

How small can we make δ as a function of Δ and n?



Δ-linear: pointwise close to linear up to $\pm\Delta$

Results for learning Δ-linear set functions

[Chierichetti et al.]:

- A randomized algorithm making $O(n^2 \log n)$ nonadaptive queries and achieving $\delta \leq O(\Delta\sqrt{n})$ w.h.p.

Our result [similar technique as Dworj-Yekhanin'08]:

- A deterministic algorithm making $O(n)$ nonadaptive queries and achieving $\delta \leq O(\Delta\sqrt{n})$.

Lower bound [Chierichetti et al.]: $\delta \geq \Omega(\Delta\sqrt{\frac{n}{\log n}})$ even with adaptivity.

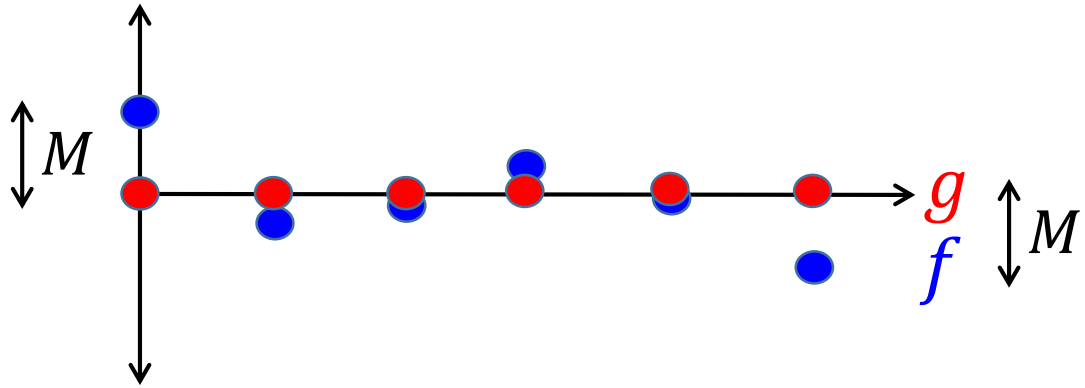
→ Algorithm useful when f is very close to linear (e.g. rounding errors).

Proof Overview

Δ-linear: pointwise close to linear up to $\pm\Delta$
ε-modular: satisfies modularity eqs. up to $\pm\epsilon$
Theorem: $\Delta < 13\varepsilon$

What we want to show

W.l.o.g. f is 1-modular and has $g = 0$ as its closest linear function.



Goal: Bound extreme value M $(= \Delta)$.

Main proof steps

1. Define a way to rearrange a collection $\mathcal{S}$ of sets: "split-and-merge"
2. Split-and-merge $\mathcal{S} \subseteq \{S | f(S) = M\}$ to bound M using 1-modularity
3. Wait! Can we split-and-merge $\mathcal{S}$?
   - Yes if $\mathcal{S}$ is sparse by existence of bipartite expander graphs
4. Construct sparse $\mathcal{S}$
5. Optimize construction to improve bound on M

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Step 1: (r, θ)-split-and-merge

The wedding planner's problem:

- k invited families, only θk tables ($\theta < 1$).
- Each person has trait: "funny", "shy", …
  - Traits do not repeat within family.
- rk seating cards ($r > 1$) direct family members to their table.

An (r, θ)-split-and-merge of the k families exists if there are seating cards s.t. traits not repeated within the same table.

Smaller r, θ → problem more constrained.

Example of (r, θ)-split-and-merge

$k = 3$ families, $rk = 6$ seating cards, $\theta k = 2$ tables

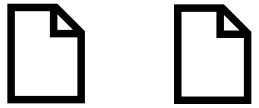
(1,2,3,4,7)

(1,2,5,6)

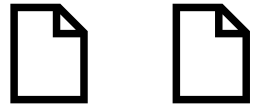
(3,4,5,7)

(1,2,7) (3,4)

(1,2) (5,6)



(3,4) (5,7)

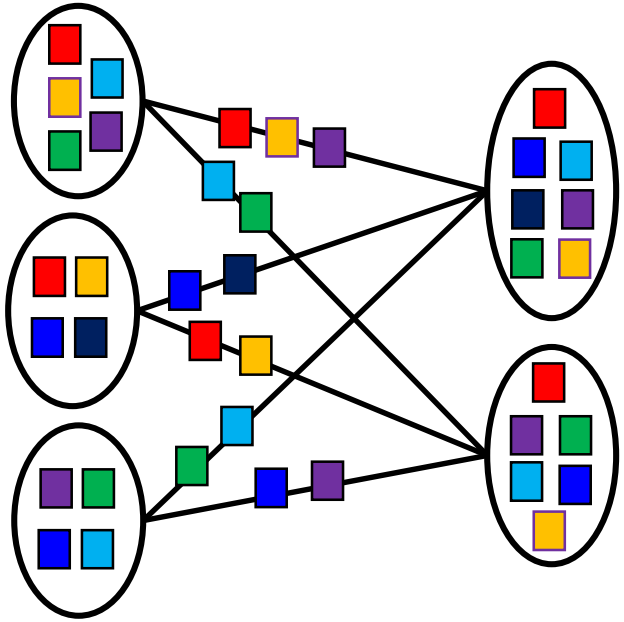


(1,2,7,5,6,3,4)

(3,4,1,2,5,7)

Split-and-merge as a bipartite graph

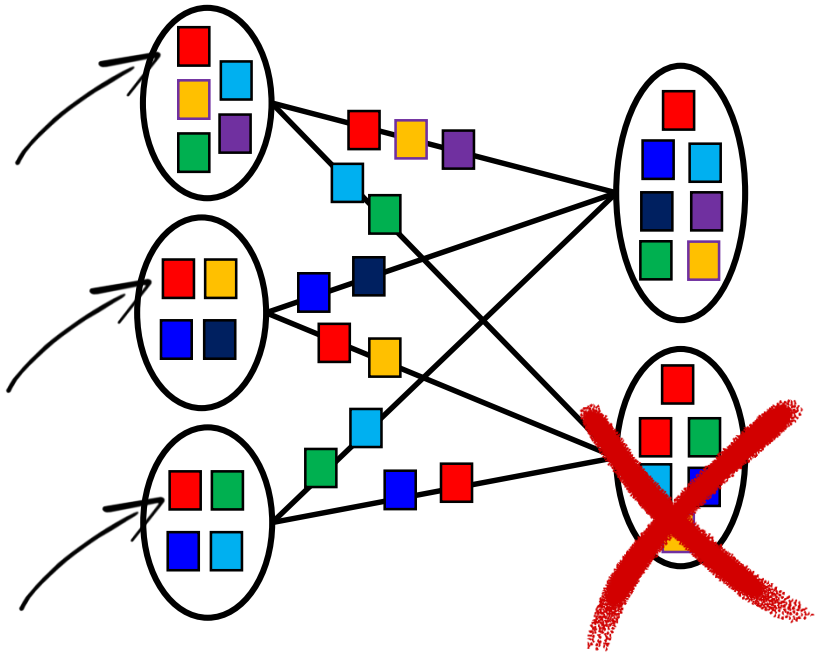
k source sets



θk target sets

rk intermediate sets

Split-and-merge not always possible



Problem: Source sets not sparse.

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1-modularity implication

Let S be disjoint union of r sets $I_1, \ldots, I_r$.

By 1-modularity of f:

$$\sum_{j \le r} f(I_j) = f(S) \pm (r-1).$$

Step 2: Applying split-and-merge to bound M

Idea: Use intermediate sets to relate values of source, target sets

Source set S is split to intermediate sets $I_1, \ldots, I_r$:

$$\sum_{j \le r} f(I_j) \ge f(S) - r + 1. \ (*)$$

Target set T is merged from intermediate sets $I_1, \ldots, I_{r/\theta}$:

$$\sum_{j \le r/\theta} f(I_j) \le f(T) + r/\theta - 1. \ (**)$$

Summing up (*), (**) over all source and target sets:

$$\sum_{j \le k} f(S_j) - rk + k \le \sum_{j \le \theta k} f(T_j) + rk - \theta k.$$

Step 2: Applying split-and-merge to bound M

Applying to source sets of value $f(S_j) = M$:

$$\rightarrow kM - rk + k \leq \theta kM + rk - \theta k \rightarrow \boxed{M \leq \frac{2r - 1 - \theta}{1 - \theta}}.$$

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Step 3: α-sparsity ensures split-and-merge

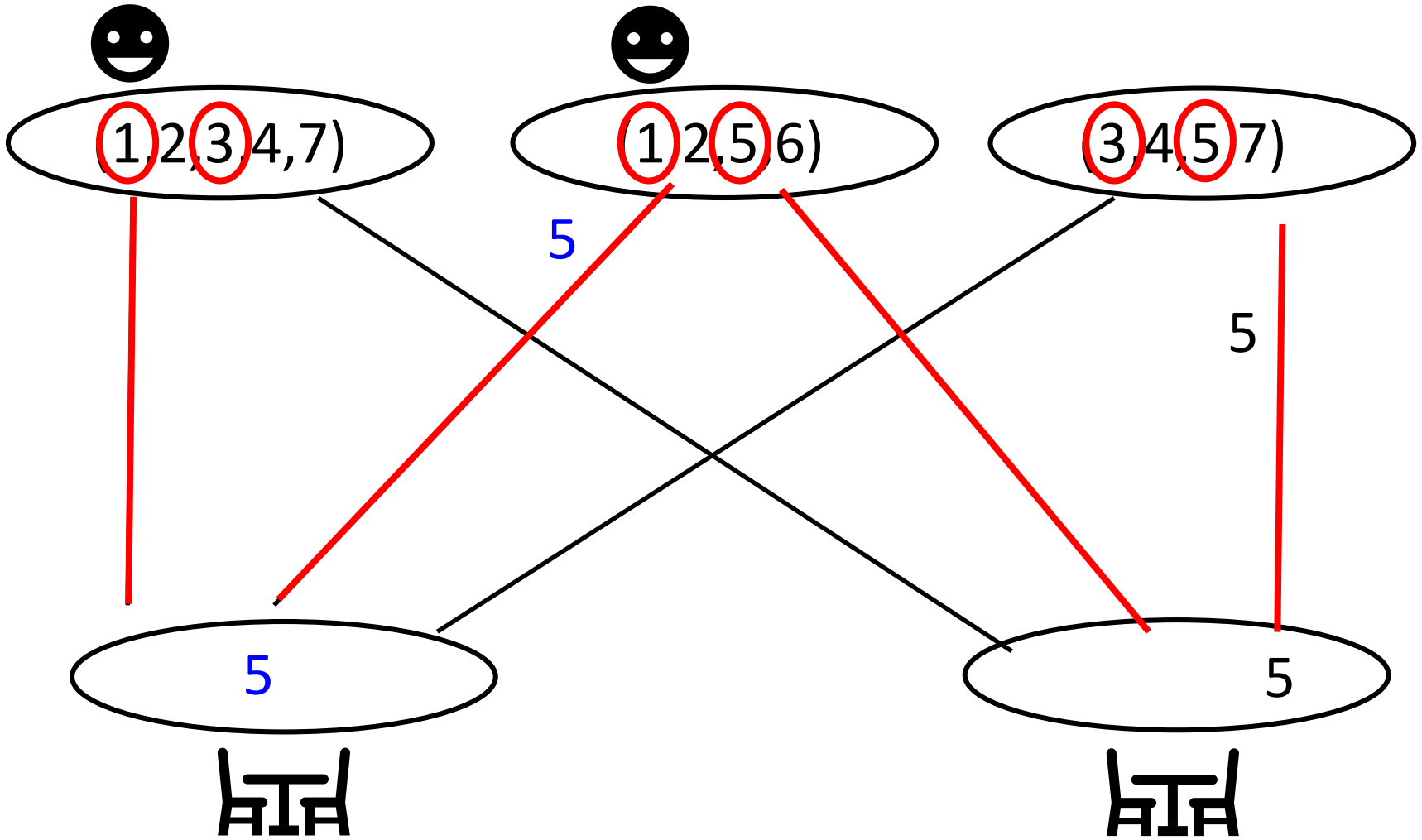
Let $\alpha < 1$.

A collection of k sets is α-sparse if each item appears in $\leq \alpha k$ sets.

Lemma [Kalton and Roberts, 1983]:

Every α-sparse collection has an (r, θ)-split-and-merge, where r, θ depend on parameters of bipartite expander graphs with α-expansion.

Example

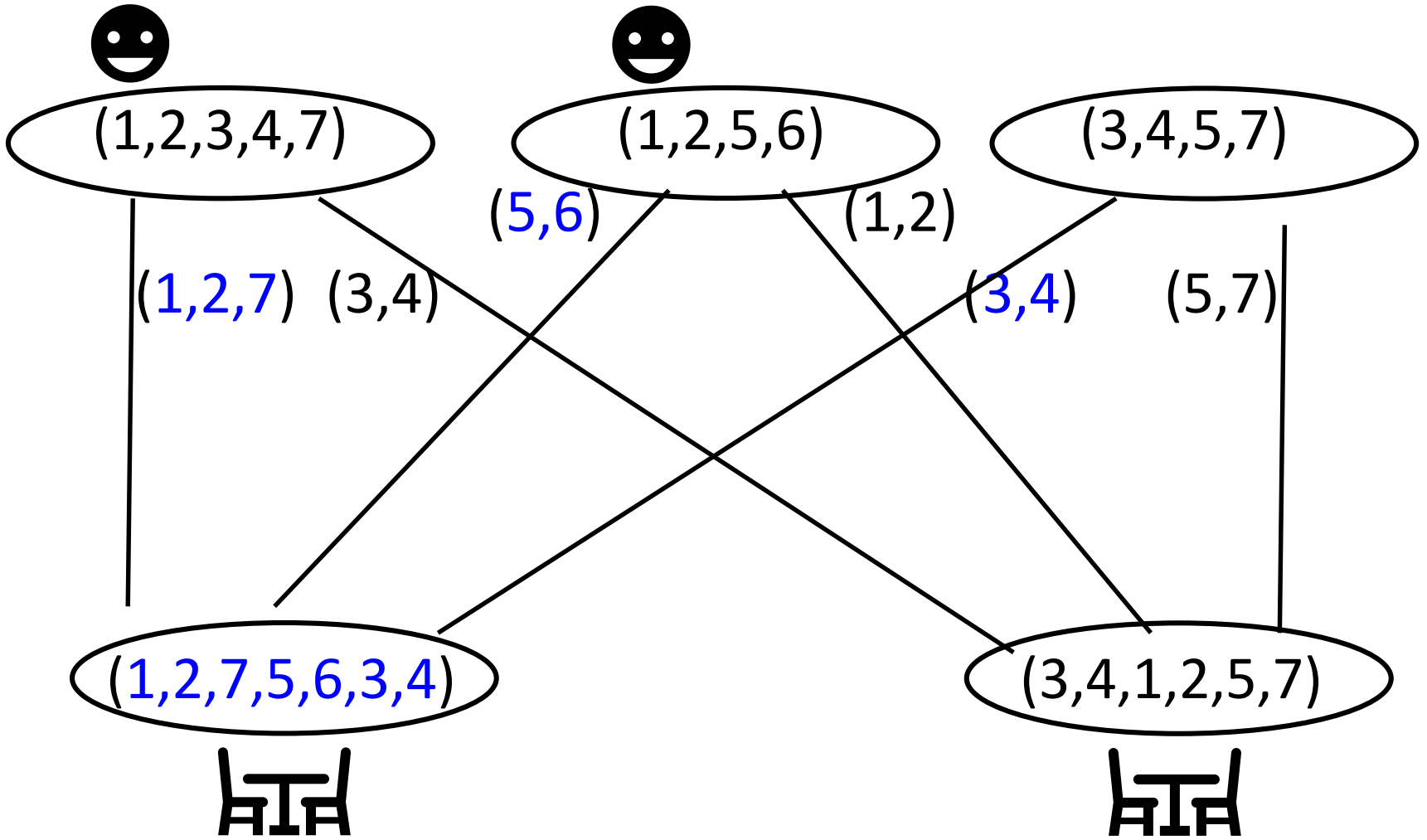


$\frac{2}{3}$-sparse collection

Expander graph with rk edges:

Every set with $\leq \frac{2}{3}$ of the k top vertices has a matching to the θk bottom vertices.

Example



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Step 4: α-sparse collection with value $\approx M$

Lemma: Every 1-modular function f whose closest linear function is 0 has a ½-sparse collection with average value $M - d$ s.t. deficit $d \leq 0.5$.

Shown by formulating an LP for closest linear function to f [Chierichetti et al, 2015], and constructing the collection from the dual distributions.

Δ-linear: pointwise close to linear up to $\pm\Delta$
ε-modular: satisfies modularity eqs. up to $\pm\epsilon$

Summary

Relating two notions of "close to" linear functions:

- Every ε-modular set function is Δ-linear for $\Delta \leq O(\varepsilon)$.
- Current bounds are $\varepsilon \leq \Delta < 13\varepsilon$.
- Proof based on expander graphs.

Learning (really) close to linear functions

- A linear function h that is $O(\Delta\sqrt{n})$-close to a Δ-linear function f can be learned by making $O(n)$ value queries non-adaptively.
- Nearly best possible.